Revisiting the inner and outer bounds for the two receiver broadcast channel

Chandra Nair

The Chinese University of Hong Kong (CUHK)

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Outline of talk

- An observation and a thought experiment
- Existing bounds
- A comparison between them
- A different way of thinking
- What is missing...
- More examples

Some preliminaries

Recall: Superposition coding can be used to achieve the union of rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq I(U;Y_1) \\ R_1 + R_2 &\leq I(U;Y_1) + I(X;Y_2|U) \\ R_1 + R_2 &\leq I(X;Y_2) \end{aligned}$$

over all $p(u,x)$.

Korner-Marton and El Gamal established that the union of rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq I(U;Y_1) \\ R_1 + R_2 &\leq I(U;Y_1) + I(X;Y_2|U) \\ R_2 &\leq I(X;Y_2) \end{aligned}$$

over all $p(u,x)$ forms an outer bound to the capacity region.

A thought experiment

Observation 1: The above inner and outer bounds seem great for a degraded scenario (where Y_1 is the weaker receiver).

Observation 2: All the capacity regions are established by showing that these two regions coincide.

Question: Are the two regions (inner and outer bounds) the same or are the different?

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Using Observation 1, a natural antipodal setting seems to be when there is no degradedness in the picture

Non-degradable BC

A non-degradable broadcast channel is one where there does not exist a non-trivial decomposition of the form

$$X \rightarrow \tilde{X} \rightarrow Y_1, Y_2$$

If

- $P : X \mapsto Y_1$
- $Q : X \mapsto Y_2$

then there does not exists M, a non-trivial $|X| \times |X|$ stochastic matrix such that

- $P = P_1 \times M; \quad Q = Q_1 \times M$

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Thus intuitively, BSSC is a perfect channel to compare the bounds

The binary skew-symmetric broadcast channel (BSSC)

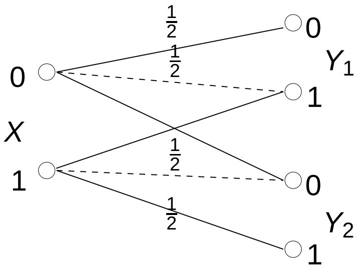

Figure: Binary Skew Symmetric Channel

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An achievable region (Marton '79)

Most of this talk, assume $R_0 = 0$ (no common message)

Recall that the following rates are achievable

$$\begin{aligned} R_1 &\leq I(U, W; Y_1) \\ R_2 &\leq I(V, W; Y_2) \\ R_1 + R_2 &\leq \min\{I(W; Y_1), I(W; Y_2)\} + I(U; Y_1|W) \\ &\quad + I(V; Y_2|W) - I(U; V|W) \end{aligned}$$

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\end{aligned}$$

This is the best achievable region known to-date

- Not even a special carefully constructed channel where one can beat this
- Obviously, no proof of optimality

Outer bound: El Gamal (Asilomar '76, IT '79)

Paper: Capacity of a class of broadcast channels (more capable)

The union of rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq I(U; Y_1) \\ R_2 &\leq I(V; Y_2) \\ R_1 + R_2 &\leq I(U; Y_1) + I(X; Y_2|U) \\ R_1 + R_2 &\leq I(V; Y_2) + I(X; Y_1|V) \end{aligned}$$

over all $p(u, v, x)$ constitutes an outer bound.

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over all $p(u, v, x)$ constitutes an outer bound.

Remark: Because this bound was not explicitly stated, this was not well-known (registered)

Call this bound the UV-OB.

Körner-Márton (IT '79)

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over all $p(v, x)$ constitutes an outer bound.

Remark: This was used in establishing the capacity of the semi-deterministic broadcast channel

Körner-Márton outer bound

Let $\mathcal{R}_a$ be the union of rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq I(X;Y_1) \\ R_2 &\leq I(V;Y_2) \\ R_1 + R_2 &\leq I(V;Y_2) + I(X;Y_1|V) \end{aligned}$$

over all $p(v,x)$.

Let $\mathcal{R}_b$ be the union of rate pairs (R_1, R_2) satisfying

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The region $\mathcal{R}_a \cap \mathcal{R}_b$ became known as the Körner-Márton outer bound.

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The following comparisons are immediate:

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Further KM-OB matches the capacity region in all special cases where capacity was established.

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The union of rate pairs (R_1, R_2) satisfying

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Also showed that

- this bound $\subseteq$ UV-OB $\subset$ KM-OB
- BSSC: UV-OB $\subset$ KM-OB (Surprise (☺))

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Also showed that

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However Nair-Wang ('08) showed that <u>the above bound</u> $\equiv$ UV-OB

Comparing inner and outer bounds

- KM-OB = Marton Inner Bound (MIB) in all special cases where capacity was established.
- UV-OB $\subset$ KM-OB
  - This implies that KM-OB $\neq$ UV-OB

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- KM-OB = Marton Inner Bound (MIB) in all special cases where capacity was established.
- UV-OB $\subset$ KM-OB
  - This implies that KM-OB $\neq$ UV-OB
- Is it true that $UV - OB = MIB$?

To, answer this we again look at the BSSC

BSSC: Comparing the bounds

Conjecture: [N-Wang '08] For all $(U, V) \to X \to (Y_1, Y_2)$

$$I(U; Y_1) + I(V; Y_2) - I(U; V) \leq \max\{I(X; Y_1), I(X; Y_2)\}$$

If the conjecture is true

- Maximum $R_1 + R_2$ achievable by Märton's strategy is 0.3616..
- Maximum $R_1 + R_2$ contained in the outer bound is 0.3725.. (N-EG '07)
- Thus inner and outer bound regions differ (!)

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The conjecture is true when $\mathrm{P}(X = 0) \in [0, \frac{1}{5}] \cup [\frac{4}{5}, 1]$

Recall: No cardinality bounds on auxiliary random variables

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Recall: No cardinality bounds on auxiliary random variables

[Gohari-Anantharam '09]

- Proved: sufficient to consider $|\mathcal{U}| \leq |\mathcal{X}|, |\mathcal{V}| \leq |\mathcal{X}|, X = f(U,V)$ to establish conjecture
- Proved: inner and outer bounds differ for BSSC

Sum-rate bounds for BSSC

Extending the perturbation method [Jog-Nair '09] established the conjecture, i.e.

For all $p(u,v,x)$, s/t $(U,V) \to X \to (Y_1, Y_2)$

$$I(U;Y_1) + I(V;Y_2) - I(U;V) \leq \max\{I(X;Y_1), I(X;Y_2)\}$$

This implies that sum-rate bounds of BSSC are:

- Marton's inner bound: 0.3616...
- UV-OB: 0.37255...
- KM-OB: 0.3743...

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Aside: Generalizing the arguments of [Jog-Nair '09], it is known that for all $p(u,v,x,y_1,y_2)$, s/t $(U,V) \to X \to (Y_1, Y_2)$

$$I(U;Y_1) + I(V;Y_2) - I(U;V) \leq \max\{I(X;Y_1), I(X;Y_2)\}$$

as long as $|X| = 2$

Open question

What is the optimal sum-rate of BSSC.

Answering this will determine whether:

- Which bound/s are loose

Possibly require new ideas

Other bounds

- Liang, Liang-Kramer had concurrently developed similar outer bounds
  - Not known if they were <u>better</u> than existing bounds (e.g., KM-OB)
- Liang, Kramer, Shamai developed the <u>New-Jersey outer bound</u> ('08)
- Nair developed another outer bound ('08). <u>No-sum-rate outer bound</u>

The following relations were established:

- No sum-rate outer bound $\subseteq$ New-Jersey outer bound
- New-Jersey outer bound $\subseteq \Big($ outer bound (Nair-El Gamal) $\cap$ outer-bound(Liang, Liang-Kramer) $\Big)$

Remark: Equivalences or strict inclusions are not established

New-Jersey outer bound (LKS '08)

The union of rate triples (R_0, R_1, R_2) satisfying

$$
\begin{aligned}
R_0 &\le \min\{I(T;Y|W_1), I(T;Z|W_2)\} \\
R_1 &\le I(U;Y|W_1) \\
R_2 &\le I(V;Z|W_2) \\
R_0 + R_1 &\le I(T,U;Y|W_1) \\
R_0 + R_1 &\le I(U;Y|T,W_1,W_2) + I(T,W_1;Z|W_2) \\
R_0 + R_2 &\le I(T,V;Z|W_2) \\
R_0 + R_2 &\le I(V;Z|T,W_1,W_2) + I(T,W_2;Y|W_1) \\
R_0 + R_1 + R_2 &\le I(U;Y|T,V,W_1,W_2) + I(T,V,W_1;Z|W_2) \\
R_0 + R_1 + R_2 &\le I(V;Z|T,U,W_1,W_2) + I(T,U,W_2;Y|W_1) \\
R_0 + R_1 + R_2 &\le I(U;Y|T,V,W_1,W_2) + I(T,W_1,W_2;Y) + I(V;Z|T,W_1,W_2) \\
R_0 + R_1 + R_2 &\le I(V;Z|T,U,W_1,W_2) + I(T,W_1,W_2;Z) + I(U;Y|T,W_1,W_2)
\end{aligned}
$$

for some $p(u)p(v)p(t)p(w_1,w_2|u,v,t)p(x|u,v,t,w_1,w_2)p(y,z|x)$ constitutes an outer bound.

An equivalent evaluatable region

The union of rate triples (R_0, R_1, R_2) satisfying

$$\begin{aligned} R_0 &\leq \min\{I(W;Y), I(W;Z)\} \\ R_0 + R_1 &\leq I(U;Y|W) + \min\{I(W;Y), I(W;Z)\} \\ R_0 + R_2 &\leq I(V;Z|W) + \min\{I(W;Y), I(W;Z)\} \\ R_0 + R_1 + R_2 &\leq \min\{I(W;Y), I(W;Z)\} + I(U;Y|W) + I(X;Z|U,W) \\ R_0 + R_1 + R_2 &\leq \min\{I(W;Y), I(W;Z)\} + I(V;Z|W) + I(X;Y|V,W) \end{aligned}$$

for some $p(u,v,w)p(y,z|x)$ is <u>equivalent</u> to the NJ-outer bound.

Proof idea: same as Nair-Wang ('08)

- Suffices to consider $|W| \leq |X| + 7; |U|, |V| \leq |X| + 2$
- If one is interested in sumrate
  - suffices to consider $|U|, |V| \leq |X|; W = \emptyset$.
- When $R_0 = 0$ this region is $\equiv$ UV-OB

Synopsis

Thus when $R_0 = 0$ we have the following current situation:

- no sum-rate outer bound $\subseteq$ UV-OB
  - No W required for the outer bound (!)
  - For inner bound, we know that W is critical even when $R_0 = 0$.

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What about no sum-rate outer bound?

How does the sum-rate of BSSC compare?

Ans: It is at least 0.37251...

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Belief: no sum-rate outer bound $\equiv$ UV-OB

Reflections

All the above outer bounds are basically algebraic manipulations that

- Start from Fano's inequality
- Use Data processing inequality
- Use Csisźar sum lemma
- Identify auxiliary random variables in terms of $M_1, M_2, Y^{i-1}, Z_{i+1}^n, etc$

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Remark: Irrespective of the algebra we do not seem to beat the UV-OB using above approach.

Hence, start from a clean slate.

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Borrows ideas and results from Images of a set by Körner-Márton ('77)

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Images of a set ...

Given $p(x)$, consider $\mathcal{B} \subset \mathcal{T}_{\epsilon}^{(n)}(\mathcal{X}^n)$

Image($\mathcal{B}$) w.r.t channel $X \mapsto Y$ is

- $\inf \frac{1}{n} \log \mathrm{P}(\mathcal{C}) : C \subseteq \mathcal{T}_{\epsilon}^{(n)}(\mathcal{Y}^n), \mathrm{P}(y^n \in \mathcal{C} | x^n) > 1 - \epsilon, \forall x^n \in \mathcal{B}$

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Remarks

- If $|B| = 1$ then $|C^*| \approx 2^{nH(Y|X)}$, and Image($\mathcal{B}$) = $-I(X;Y)$
- If $\mathcal{B}$ is a code book of size 2^{nR}, then Image($\mathcal{B}$) = $R - I(X;Y)$
- If $\mathcal{B} \neq \emptyset$, then $-I(X;Y) \leq$ Image($\mathcal{B}$) ≤ 0

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Theorem (KM-77)

If Image($\mathcal{B}$)$_{X \mapsto Y} \geq t$, then Image($\mathcal{B}$)$_{X \mapsto Z} \geq T_{Y \to Z}(t)$, where

$$T_{Y \to Z}(t) = min\{r - I(U;Z) : r - I(U;Y) \geq t, 0 \leq r \leq I(U;Y)\}$$

A reasoning

Consider a good code book (maximal error probability is small) (Willems '91)

Let $\mathcal{B}_i = \{x^n(i,j), j \in (1, ..., 2^{nR_2})\}$.

Properties

1. Each $\mathcal{B}_i$ is a 2^{nR_2} code book for receiver Z
   - Image $(\mathcal{B}_i)_{X \mapsto Z} \geq R_2 - I(X;Z)$
   - Therefore, Image $(\mathcal{B}_i)_{X \mapsto Y} \geq T_{Z \to Y}\big(R_2 - I(X;Z)\big)$

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2. The receiver Y can distinguish between $\mathcal{B}_i$, i.e. Images $(\mathcal{B}_i)_{X \mapsto Y}$ are disjoint
    - Therefore $R_1 + T_{Z \to Y}\big(R_2 - I(X;Z)\big) \leq 0$

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Thus any good codebook must satisfy

$$R_1 + T_{Z \to Y}\left(R_2 - I(X;Z)\right) \leq 0$$
$$R_2 + T_{Y \to Y}\left(R_1 - I(X;Y)\right) \leq 0 \text{ (interchange roles)}$$

Comparison

How good is the outer bound (OB)

$$R_1 + T_{Z\to Y}\big(R_2 - I(X;Z)\big) \le 0$$
$$R_2 + T_{Y\to Z}\big(R_1 - I(X;Y)\big) \le 0$$

Remarks:

- OB $\subseteq$ UV-OB

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$$R_2 + T_{Y\to Z}\big(R_1 - I(X;Y)\big) \leq 0$$

Remarks:

- OB $\subseteq$ UV-OB
- Litmus test: Sumrate of BSSC
  - Sumrate of OB (BSSC) = 0.37255.. = Sumrate of UV-OB (BSSC)
  - Fails the litmus test ☺

Comparison

How good is the outer bound (OB)

$$R_1 + T_{Z \to Y}\big(R_2 - I(X;Z)\big) \leq 0$$
$$R_2 + T_{Y \to Z}\big(R_1 - I(X;Y)\big) \leq 0$$

Remarks:

- OB $\subseteq$ UV-OB
- Litmus test: Sumrate of BSSC
  - Sumrate of OB (BSSC) = 0.37255.. = Sumrate of UV-OB (BSSC)
  - Fails the litmus test ☺

Silver lining: There is another property that a good code book must have

A figure showing the issue

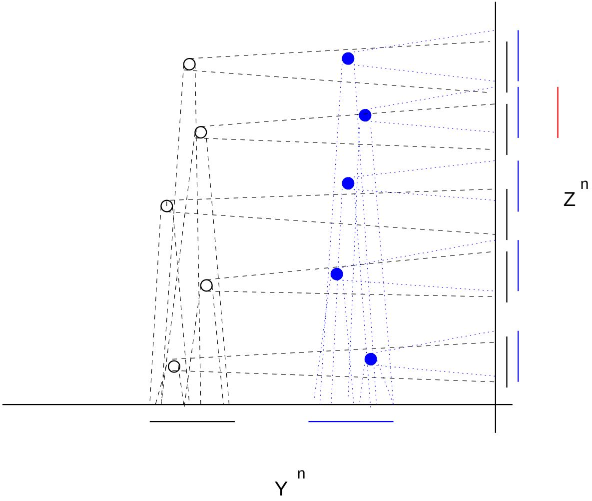

Figure: An overcounting

Remarks

- We figured a possible over counting with OB
- Do we need to bother about this over lap (over-counting)
  - No - degraded, less noisy, more capable (superposition coding)
    - Disjoint images in weaker receiver can be made to be disjoint in stronger receiver (without losing anything in exponent)
  - No - semideterministic
    - The images on the deterministic receiver are point sets (!)

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    - The images on the deterministic receiver are point sets (!)
- Surprise: These are precisely the classes where capacity is known (!)

Therefore one needs to show either of the two:

- We need not bother with this over-counting
- This over-counting does matter and UV-OB can be made tighter.

Remarks

Looked at existing bounds

- OB (with R_0) is a simple evaluatable region
- when $R_0 = 0$, UV-OB still rules !

Introduced Litmus test ☺

- Compare the sum rate to that of BSSC

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Introduced Litmus test ☺

- Compare the sum rate to that of BSSC

Derived a new looking bound using a much more intuitive reasoning

- Showed that it is as good as UV-OB
- However litmus test failed
- Identified a possible over counting (weakness in outer bound)

Outline of talk

- Existing outer bounds
- A comparison between them
- A different way of thinking
- What is missing...
- More examples

BISO broadcast channels

BISO: (Binary-Input Symmetric-Output)
A channel is BISO if the channel transition matrix satisfies

$$\mathrm{P}(Y = k|X = 0) = \mathrm{P}(Y = -k|X = 1), \forall k$$

Examples: BSC, BEC

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[Geng-Nair-Shamai-Wang '10]

Consider a BC where $X \mapsto Y_1, X \mapsto Y_2$ are BISO channels

Then the following are equivalent:

- Neither is more capable than the other, i.e. $\exists p_1, p_2$ s.t

$$I(X;Y_1) > I(X;Y_2)|_{\mathrm{P}(X=0)=p_1}, \quad I(X;Y_1) < I(X;Y_2)|_{\mathrm{P}(X=0)=p_2}.$$

- Marton's inner bound $\subset$ UV-OB

There are BISO broadcast channels with $|Y| \geq 4$ which are not more-capable comparable

Thank You

More on Thursday